

# ENG 301

## Engineering Mathematics III

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# LINEAR ALGEBRA

## SYSTEMS OF LINEAR EQUATIONS

WRITING SYSTEMS  $20X + 35Y = 70$

$$\begin{aligned} x_1 + x_2 &= 5 \\ 3x_1 - 2x_2 &= 10 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$$

Coefficient matrix

$$\left[ \begin{array}{cc|c} 1 & 1 & 5 \\ 3 & -2 & 10 \end{array} \right]$$

Augmented Matrix

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WRITING SYSTEMS

$$\begin{aligned} 2x_1 + x_2 &= 5 \quad 10 \\ + 3x_1 - 2x_2 &= 10 \quad * \end{aligned}$$

$$\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$$

Coefficient matrix

$$\left[ \begin{array}{cc|c} 1 & 1 & 5 \\ 3 & -2 & 10 \end{array} \right]$$

Augmented Matrix

$$\begin{aligned} 5x_1 &= 20 \\ 5 & \quad 5 \end{aligned}$$

$$x_1 = 4$$

$$\begin{aligned} x_1 + x_2 &= 5 \\ 4 + x_2 &= 5 \\ x_2 &= 1 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 5 \\ 3 & -2 & 10 \end{bmatrix} \xrightarrow{2R_1 \rightarrow R_2} \begin{bmatrix} 5 & 0 & 20 \\ 3 & -2 & 10 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{5}R_1 \rightarrow R_1} \begin{bmatrix} 1 & 0 & 4 \\ 3 & -2 & 10 \end{bmatrix} \xrightarrow{-3R_1 \rightarrow R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & -2 & -2 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_2 \rightarrow R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{aligned} 1x_1 + 0x_2 &= 4 & x_1 &= 4 \\ 0x_1 + 1x_2 &= 1 & x_2 &= 1 \end{aligned}$$

WRITING SYSTEMS

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## TERMINOLOGY

**LINEAR EQUATION** - An equation that can be written as  $a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b$  where  $a_1, a_2, \dots, a_n, b$  are real or complex numbers.

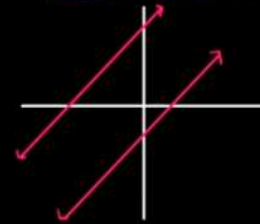
**SYSTEM OF LINEAR EQUATIONS** - A collection of two or more linear equations using the same variables.

**SOLUTION** - A list of numbers  $(s_1, s_2, s_3, \dots)$  that makes each equation in the system true when substituted for  $x_1, x_2, x_3, \dots$  respectively.

**SOLUTION SET** - The set of all possible solutions to a system.

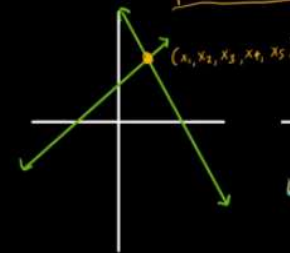
## TYPES OF SYSTEMS

**INCONSISTENT**



★ NO SOLUTION

**CONSISTENT**



ONE SOLUTION

TWO LINES THAT ARE IDENTICAL



INFINITELY MANY SOLUTIONS

## PRACTICE

SOLVE THE GIVEN SYSTEM OF EQUATIONS USING ELIMINATION

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

$(29, 16, 3)$

## PRACTICE

SOLVE THE GIVEN SYSTEM OF EQUATIONS USING ELIMINATION.

$$\begin{aligned} + (x_1 - 2x_2 + x_3 &= 0) & \rightarrow & 2x_2 - 8x_3 = 8 \\ & 2x_2 - 8x_3 = 8 \\ & -4x_1 + 5x_2 + 9x_3 = -9 \end{aligned}$$

$$\begin{aligned} 4x_1 - 8x_2 + 4x_3 &= 0 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

## PRACTICE

SOLVE THE GIVEN SYSTEM OF EQUATIONS USING ELIMINATION.

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SOLVE THE SYSTEM (AGAIN)

THIS TIME USE AN AUGMENTED MATRIX AND ROW OPERATIONS

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + x_2 + x_3 &= -9 \end{aligned} \quad \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 1 & 1 & -9 \end{array} \right]$$

(29, 14, 3)

SOLVE THE SYSTEM (AGAIN)

THIS TIME USE AN AUGMENTED MATRIX AND ROW OPERATIONS

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + x_2 + x_3 &= -9 \end{aligned} \quad \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 1 & 1 & -9 \end{array} \right]$$

SOLVE THE SYSTEM (AGAIN)

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$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + x_2 + x_3 &= -9 \end{aligned} \quad \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 1 & 1 & -9 \end{array} \right] \xrightarrow{4R_1 + R_3 \rightarrow R_3} \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 5 & -9 \end{array} \right]$$

$$\xrightarrow{\frac{1}{2}R_2 \rightarrow R_2} \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 5 & -9 \end{array} \right] \xrightarrow{3R_2 + R_3 \rightarrow R_3} \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & -7 & -3 \end{array} \right]$$

$$\begin{aligned} x_1 - 2x_2 + 1x_3 &= 0 \\ 1x_2 - 4x_3 &= 4 \\ 1x_3 &= 3 \end{aligned} \quad \begin{aligned} x_1 - 2(14) + 3 &= 0 \\ x_1 - 32 + 3 &= 0 \\ x_1 - 29 &= 0 \\ x_1 &= 29 \end{aligned}$$

Back  
Subst.

(29, 14, 3)

Try to complete these solutions using the row operations from here!

Hint? ...

$$\left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & -7 & -3 \end{array} \right]$$

EXISTENCE AND UNIQUENESS

DETERMINE IF THE SYSTEM IS CONSISTENT (DOES A SOLUTION EXIST?)

IF SO, DETERMINE IF THE SOLUTION IS UNIQUE (JUST ONE SOLUTION?)

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned}$$

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EXISTENCE AND UNIQUENESS

DETERMINE IF THE SYSTEM IS CONSISTENT (DOES A SOLUTION EXIST?)  
 IF SO, DETERMINE IF THE SOLUTION IS UNIQUE (JUST ONE SOLUTION?)

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned} \quad \left[ \begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[ \begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{array} \right]$$

$$\xrightarrow{\frac{1}{2}R_1 \rightarrow R_1} \left[ \begin{array}{ccc|c} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{array} \right] \xrightarrow{+R_1 \rightarrow R_3} \left[ \begin{array}{ccc|c} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -7 \end{array} \right] \xrightarrow{2R_2 + R_3 \rightarrow R_3} \left[ \begin{array}{ccc|c} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{array} \right]$$

$$0x_1 + 0x_2 + 0x_3 = 15$$

INCONSISTENT

# LINEAR ALGEBRA

## ROW REDUCTION AND ECHELON FORMS

ECHELON FORM VS. REDUCED ROW ECHELON FORM (RREF)

ECHELON: (PREVIOUSLY CALLED TRIANGLE FORM)

- 1) ALL NON-ZERO ROWS ARE ABOVE ALL ZERO ROWS
- 2) EACH LEADING ENTRY OF A ROW IS IN A COLUMN TO THE RIGHT OF THE LEADING ENTRY OF THE ROW ABOVE IT.
- 3) ALL ENTRIES IN A COLUMN BELOW A LEADING ENTRY ARE ZEROS.

RREF - ALL CONDITIONS ABOVE AND:

- 4) THE LEADING ENTRY IN EACH NON-ZERO ROW IS 1
- 5) EACH LEADING 1 IS THE ONLY NON-ZERO ENTRY IN THE COLUMN

ECHELON FORM VS. REDUCED ROW ECHELON FORM (RREF)

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- 1) ALL NON-ZERO ROWS ARE ABOVE ALL ZERO ROWS
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- 3) ALL ENTRIES IN A COLUMN BELOW A LEADING ENTRY ARE ZEROS.

$$\left[ \begin{array}{cccc|c} 2 & 7 & 6 & 3 & 2 \\ 0 & 1 & 4 & 1 & 3 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

RREF - ALL CONDITIONS ABOVE AND:

- 4) THE LEADING ENTRY IN EACH NON-ZERO ROW IS 1
- 5) EACH LEADING 1 IS THE ONLY NON-ZERO ENTRY IN THE COLUMN

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$



## PIVOT!

$$\left[ \begin{array}{cccc|c} \boxed{1} & 4 & 2 & 3 & 9 \\ 0 & 0 & \boxed{2} & 7 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

★      ★

PIVOT POSITION - CORRESPONDS TO LEADING  $\boxed{1}$  IN RREF

PIVOT COLUMN - THE COLUMN THAT CONTAINS THE PIVOT

PIVOT - NONZERO NUMBER IN PIVOT POSITION USED TO CREATE ZEROS IN ROW OPERATIONS

## THE ROW REDUCTION ALGORITHM

1. BEGIN AT LEFTMOST NONZERO COLUMN, WHICH IS A PIVOT COLUMN. SELECT A NONZERO ENTRY AS PIVOT AND INTERCHANGE, IF NECESSARY, TO MOVE THAT ENTRY INTO THE PIVOT POSITION (Row 1).

$$\begin{aligned} x_1 - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned} \quad \left[ \begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array} \right]$$

## THE ROW REDUCTION ALGORITHM

2. USE ROW OPERATIONS TO CREATE ZEROS IN ALL ENTRIES BELOW THE PIVOT.

$$\left[ \begin{array}{ccc|c} \boxed{1} & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array} \right] \xrightarrow{-2R_1 + R_2 \rightarrow R_2} \left[ \begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{array} \right]$$

## THE ROW REDUCTION ALGORITHM

3. REPEAT THIS PROCESS FOR REMAINING ROWS, IGNORING ROWS YOU'VE ALREADY APPLIED ALGORITHM TO.

$$\left[ \begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{array} \right] \xrightarrow{\frac{1}{2}R_2 \rightarrow R_2} \left[ \begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 7.5 & -4.5 \\ 0 & 1 & 5 & -2 \end{array} \right] \xrightarrow{-R_1 + R_2 \rightarrow R_2} \left[ \begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 7.5 & -4.5 \\ 0 & 0 & 2.5 & 2.5 \end{array} \right] \xrightarrow{-\frac{2}{5}R_3 \rightarrow R_3} \left[ \begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 7.5 & -4.5 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

## THE ROW REDUCTION ALGORITHM

4. ENSURE EACH PIVOT IS A 1, USING SCALING AS NECESSARY

$$\left[ \begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 7.5 & -4.5 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

## THE ROW REDUCTION ALGORITHM

5. BEGINNING WITH THE RIGHTMOST PIVOT AND WORKING UPWARDS AND TO THE LEFT, USE ROW OPERATIONS TO CREATE ZEROS ABOVE EACH PIVOT.

$$\left[ \begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 7.5 & -4.5 \\ 0 & 0 & 1 & -1 \end{array} \right] \xrightarrow{-\frac{15}{2}R_3 + R_2 \rightarrow R_2} \left[ \begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right] \xrightarrow{3R_3 + R_1 \rightarrow R_1} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

$x_1 - 3x_3 = 8$   
 $2x_1 + 2x_2 + 9x_3 = 7$   
 $x_2 + 5x_3 = -2$

$5 - 3(-1) = 8$   
 $5 = 8$  ✓  
 $8 = 8$  ✓

$3 + 5(-1) = -2$   
 $3 - 5 = -2$   
 $-2 = -2$  ✓

$x_1 = 5$   
 $x_2 = 3$   
 $x_3 = -1$

**(5, 3, -1)**

# LINEAR ALGEBRA

SOLUTION SETS AND FREE VARIABLES

## CONSISTENT SYSTEM WITH INFINITELY MANY SOLUTIONS

$$\left[ \begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \begin{cases} x_1 - 5x_3 = 1 \\ x_2 + x_3 = 4 \\ 0 = 0 \end{cases} \rightarrow \begin{cases} x_1 = 5x_3 + 1 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases}$$

FREE VARIABLES CAN TAKE ON ANY VALUE. ONCE YOU CHOOSE A VALUE FOR YOUR FREE VARIABLE, IT WILL DETERMINE THE VALUES OF THE OTHER (BASIC) VARIABLES.

LET  $x_3 = 2$

$$x_1 = 5(2) + 1 = 10 + 1 = 11$$

$$x_2 = 4 - 2 = 2$$

$$(11, 2, 2)$$

LET  $x_3 = -6$

$$x_1 = 5(-6) + 1 = -30 + 1 = -29$$

$$x_2 = 4 - (-6) = 10$$

$$(-29, 10, -6)$$

## FIND THE GENERAL SOLUTION TO THE SYSTEM

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2 \end{aligned} \quad \left[ \begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{array} \right] \xrightarrow{\substack{2R_1 + R_2 \\ -3R_1 + R_3}} \left[ \begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{array} \right]$$

$$\xrightarrow{R_2 + R_3} \left[ \begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & -3 & -1 & 2 \end{array} \right] \quad 0 \neq 5 \text{ INCONSISTENT}$$

## PRACTICE

### FIND THE GENERAL SOLUTION OF THE AUGMENTED MATRIX

$$\left[ \begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{array} \right] \xrightarrow{-3R_1 + R_2} \left[ \begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{array} \right] \xrightarrow{-1/5 R_2} \left[ \begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$\star x_1 + 3x_2 + 4x_3 = 7$$

$$x_3 = 3$$

$$x_1 + 3x_2 + 4(3) = 7$$

$$x_1 + 3x_2 + 12 = 7$$

$$x_1 + 3x_2 = -5$$

$$x_1 = -5 - 3x_2$$

$$\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$$

# LINEAR ALGEBRA

## VECTOR EQUATIONS

### VECTORS IN $\mathbb{R}^2$

$$(1, 3) \rightarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

**VECTOR** - AN ORDERED LIST OF NUMBERS (MORE ON THIS IN CH. 4)

**COLUMN VECTOR** - A VECTOR WITH ONLY ONE COLUMN. WE OFTEN USE THESE FOR ORDERED PAIRS, TRIPLES, ETC.

**VECTORS IN  $\mathbb{R}^2$**  - THE SET OF ALL VECTORS WITH 2 ENTRIES.  
 $\mathbb{R}$  → REAL NUMBERS    2 → NUMBER OF ENTRIES  
 THIS IS THE SET OF ALL POINTS IN A PLANE.

**OPERATIONS WITH VECTORS** - SAME AS WITH OTHER MATRICES.

$$2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

**SCALAR** - MULTIPLY VECTOR BY A CONSTANT

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

**ADDITION** - ADD CORRESPONDING VALUES

**MULTIPLICATION** - NOPE! DIMENSIONS DON'T WORK.

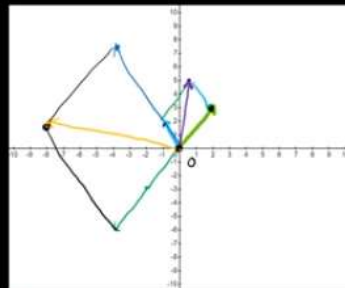
### OPERATIONS ON VECTORS EXAMPLE

IF  $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  AND  $\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ , FIND  $\vec{u} + \vec{v}$  AND  $-2\vec{u} + 4\vec{v}$ .

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$-2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -4 \\ -6 \end{bmatrix} + \begin{bmatrix} -4 \\ 8 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \end{bmatrix}$$



PARALLELOGRAM RULE FOR ADDITION ★

### VECTORS IN $\mathbb{R}^n$

IF  $n \in \mathbb{R}$ , THEN  $\mathbb{R}^n$  IS THE COLLECTION OF ALL LISTS OF ORDERED  $n$ -TUPLES OF  $n$  REAL NUMBERS WRITTEN AS  $n \times 1$  COLUMN MATRICES.

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}$$

$$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

**ZERO VECTOR** - THE VECTOR WHOSE ENTRIES ARE ALL 0, DENOTED BY  $\vec{0}$ .

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



## ALGEBRAIC PROPERTIES OF $\mathbb{R}^n$

THESE PROPERTIES CORRESPOND TO PROPERTIES OF REAL NUMBERS AND PERTAIN TO VECTORS  $u, v$  AND  $w$  AND SCALARS  $c$  AND  $d$ .

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

$$\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$$

$$\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}$$

$$c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$$

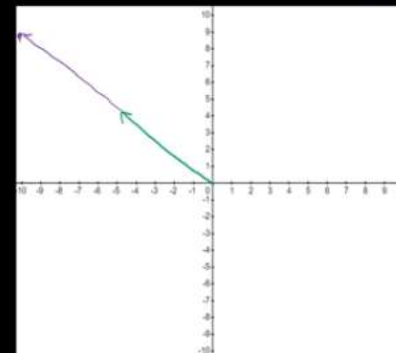
$$(c+d)\vec{u} = c\vec{u} + d\vec{u}$$

$$c(d\vec{u}) = (cd)\vec{u}$$

$$1\vec{u} = \vec{u}$$

## PRACTICE

LET  $\vec{u} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$ . DISPLAY THE VECTORS  $\vec{u}$ ,  $2\vec{u}$  AND  $\frac{1}{2}\vec{u}$  ON A GRAPH.



## SOME CLARIFICATION

A VECTOR EQUATION  $x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{b}$  HAS THE SAME SOLUTION SET AS THE LINEAR SYSTEM WHOSE AUGMENTED MATRIX IS  $[\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n \ | \ \vec{b}]$ . THEREFORE, A VECTOR EQUATION ONLY HAS A SOLUTION IF THE SYSTEM IS CONSISTENT.

IF  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$  ARE IN  $\mathbb{R}^n$ , THEN THE SET OF LINEAR COMBINATIONS IS DENOTED  $\text{SPAN}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$  AND IS CALLED THE SUBSET OF  $\mathbb{R}^n$  SPANNED. ESSENTIALLY

$\text{SPAN}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$  IS ALL VECTORS THAT CAN BE WRITTEN IN THE FORM  $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p = \vec{b}$  WITH  $c_i$  SCALARS.

# LINEAR ALGEBRA

LINEAR COMBINATIONS

### LINEAR COMBINATIONS EXISTENCE

IF  $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$  AND  $\vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ , DETERMINE WHETHER  $b = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$  CAN BE WRITTEN AS A LINEAR COMBINATION OF  $v_1$  AND  $v_2$ , THEN DETERMINE THE WEIGHTS SUCH THAT  $c_1 v_1 + c_2 v_2 = b$ .

$$c_1 v_1 + c_2 v_2 = b$$

$$c_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 & 2c_2 \\ -2c_1 & 5c_2 \\ -5c_1 & 6c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \rightarrow \left[ \begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array} \right]$$

$$\left[ \begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array} \right] \xrightarrow{2R_1 + R_2, 5R_1 + R_3} \left[ \begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 18 \\ 0 & 16 & 32 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[ \begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 16 & 32 \\ 0 & 1 & 18 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[ \begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 18 \\ 0 & 16 & 32 \end{array} \right] \xrightarrow{-16R_2 + R_3} \left[ \begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 18 \\ 0 & 0 & 0 \end{array} \right]$$

$$\frac{1}{c_1} = \frac{3}{2} \quad 3 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \checkmark$$

### MORE SPAN TALK

IS  $b$  IN THE PLANE CREATED BY  $\text{SPAN}\{a_1, a_2\}$  IF

$$\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix} \text{ AND } \vec{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

$$\left[ \begin{array}{cc|c} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{array} \right] \xrightarrow{2R_1 + R_2, -3R_1 + R_3} \left[ \begin{array}{cc|c} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{array} \right] \xrightarrow{-6R_2 + R_3} \left[ \begin{array}{cc|c} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{array} \right]$$

$$0 \neq -2$$

NO SOL.

### PRACTICE

A MINING COMPANY HAS TWO MINES. ONE DAY'S OPERATION AT MINE 1 PRODUCES ORE THAT CONTAINS 20 METRIC TONS OF COPPER AND 550 KG OF SILVER. MINE 2 PRODUCES 30 METRIC TONS OF COPPER AND 500 KG OF SILVER. HOW MANY DAYS SHOULD EACH MINE OPERATE TO PRODUCE 150 TONS COPPER AND 2825 KG SILVER

$$x_1 \begin{bmatrix} 20 \\ 550 \end{bmatrix} + x_2 \begin{bmatrix} 30 \\ 500 \end{bmatrix} = \begin{bmatrix} 150 \\ 2825 \end{bmatrix}$$

$$\left[ \begin{array}{cc|c} 20 & 30 & 150 \\ 550 & 500 & 2825 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 1.5 & 7.5 \\ 550 & 500 & 2825 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 1.5 & 7.5 \\ 0 & -325 & -1300 \end{array} \right]$$

$$\sim \left[ \begin{array}{cc|c} 1 & 1.5 & 7.5 \\ 0 & 1 & 4 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 0 & 1.5 \\ 0 & 1 & 4 \end{array} \right]$$

MINE 1 1.5 days  
MINE 2 4 days

### PRACTICE 2

LET  $a_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$ ,  $a_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$  AND  $b = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$ . FOR WHAT VALUE(S) OF  $h$  IS  $b$  IN THE SPAN OF  $a_1$  AND  $a_2$ ?

$$\left[ \begin{array}{cc|c} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & 8+h \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 3 & 8+h \end{array} \right]$$

$$\left[ \begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 17+h \end{array} \right]$$

$0 = 0$

$$17+h=0$$

$$h=-17$$

$h$  must be -17 for  $b$  to be in the span  $\{a_1, a_2\}$

# LINEAR ALGEBRA

THE MATRIX EQUATION  $Ax = b$

## LINEAR COMBINATIONS AS THE PRODUCT OF A MATRIX AND VECTOR

IF  $A$  IS AN  $m \times n$  MATRIX WITH COLUMNS  $a_1, a_2, \dots, a_n$  AND IF  $x \in \mathbb{R}^n$ , THEN  $Ax$  IS THE LINEAR COMBINATION OF THE COLUMNS OF  $A$  USING THE CORRESPONDING ENTRIES IN  $x$  AS WEIGHTS:

$$A_x = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

Diagram illustrating the matrix multiplication  $Ax$ . The matrix  $A$  is shown as a row of columns  $a_1, a_2, a_3, \dots, a_n$ . The vector  $x$  is shown as a column of entries  $x_1, x_2, x_3, \dots, x_n$ . The result is a linear combination of the columns of  $A$  weighted by the entries in  $x$ .

## EXAMPLES

1. FIND  $Ax$  IF  $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$  AND  $x = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ .

$$\begin{aligned} Ax &= 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 4 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 6 \end{bmatrix} + \begin{bmatrix} -3 \\ 12 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 17 \end{bmatrix} \end{aligned}$$

2. WRITE THE LINEAR COMBINATION  $2v_1 - 3v_2 + 4v_3$  AS A MATRIX TIMES A VECTOR.  $Ax$

$$Ax = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

## THE SAME - BUT DIFFERENT

SYSTEM OF EQUATIONS:

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 3 \\ 2x_2 + 3x_3 &= 4 \end{aligned}$$

AUGMENTED MATRIX:

$$\left[ \begin{array}{ccc|c} 2 & 3 & -1 & 3 \\ 0 & 2 & 3 & 4 \end{array} \right]$$

VECTOR EQUATION:  $x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots + x_n \vec{a}_n = \vec{b}$

$$x_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

MATRIX EQUATION:  $Ax = b$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$